AQA

Please write clearly in block capitals.

Centre number	
Candidate number	
Surname	
Forename(s)	
Candidate signature	

AS
FURTHER MATHEMATICS

Paper 1 **MODEL ANSWERS**

Monday 14 May 2018 Afternoon Time allowed: 1 hour 30 minutes

Materials
- You must have the AQA formulae and statistical tables booklet for A-level Mathematics and A-level Further Mathematics.
- You should have a scientific calculator that meets the requirements of the specification. (You may use a graphical calculator.)

Instructions
- Use black ink or black ball-point pen. Pencil should only be used for drawing.
- Fill in the boxes at the top of this page.
- Answer **all** questions.
- You must answer each question in the space provided for that question. If you require extra space for your answer(s), use the lined pages at the end of this book. Write the question number against your answer(s).
- Do **not** write outside the box around each page.
- Show all necessary working; otherwise marks for method may be lost.
- Do all rough work in this book. Cross through any work you do not want to be marked.

Information
- The marks for questions are shown in brackets.
- The maximum mark for this paper is 80.

Advice
- Unless stated otherwise, you may quote formulae, without proof, from the booklet.
- You do not necessarily need to use all the space provided.

For Examiner's Use	
Question	Mark
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Answer **all** questions in the spaces provided.

1 $z = 3 - \mathrm{i}$

Determine the value of zz^*

Circle your answer.

[1 mark]

10 $\quad\sqrt{10}\quad$ $10 - 2\mathrm{i}$ $\quad 10 + 2\mathrm{i}$

$z = 3 - i \quad z^* = 3 + i$

$zz^* = (3-i)(3+i) = 3^2 - i^2 = 9 + 1 = 10$

2 Three matrices **A**, **B** and **C** are given by

$$\mathbf{A} = \begin{bmatrix} 5 & 2 & -3 \\ 0 & 7 & 6 \\ 4 & 1 & 0 \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} 1 & 0 \\ 3 & -5 \\ -2 & 6 \end{bmatrix} \qquad \text{and } \mathbf{C} = \begin{bmatrix} 6 & 4 & 3 \\ 1 & 2 & 0 \end{bmatrix}$$

Which of the following **cannot** be calculated?

Circle your answer.

[1 mark]

AB **AC** **BC** $\mathbf{A}^2$

The number of columns in the first matrix must be the same as the number of rows in the next

3 Which of the following functions has the fourth term $-\frac{1}{720}x^6$ in its Maclaurin series expansion?

Circle your answer.

[1 mark]

$\sin x$ $\quad\cos x\quad$ e^x $\quad\ln(1 + x)$

From formula booklet, the n^{th} term of the Maclaurin series is: $(-1)^n \frac{x^{2n}}{(2n)!}$

which evaluates to $-\frac{1}{720}x^6$ when $n=6$

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4 Sketch the graph given by the polar equation

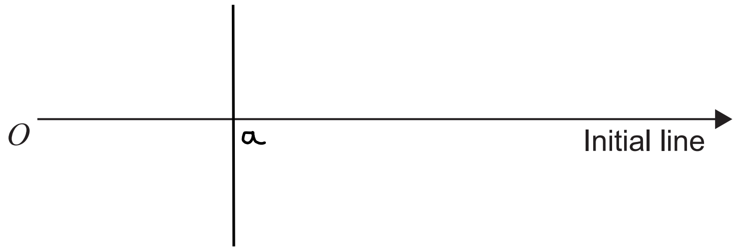
$$r = \frac{a}{\cos\theta}$$

where a is a positive constant.

[2 marks]

$r = \frac{a}{\cos\theta} \Rightarrow r\cos\theta = a$

Polar coordinates so $x = r\cos\theta$ and thus $x = a$ so we have a straight vertical line



5 Describe fully the transformation given by the matrix $\begin{bmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$

[3 marks]

The matrix is in the form $\begin{pmatrix} c & -s & 0 \\ s & c & 0 \\ 0 & 0 & 1 \end{pmatrix}$ where $c = \cos\theta$ and $s = \sin\theta$

This means $\cos\theta = -\frac{1}{2}$ and $\sin\theta = \frac{\sqrt{3}}{2} \Rightarrow \theta = 120^\circ$

Therefore, the matrix represents a rotation though 120° about the z-axis

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6 (a) Matthew is finding a formula for the inverse function arsinh x.
He writes his steps as follows:

$$\text{Let } y = \sinh x$$

$$y = \frac{1}{2}(e^x - e^{-x})$$

$$2y = e^x - e^{-x}$$

$$0 = e^x - 2y - e^{-x}$$

$$0 = (e^x)^2 - 2ye^x - 1$$

$$0 = (e^x - y)^2 - y^2 - 1$$

$$y^2 + 1 = (e^x - y)^2$$

$$\pm\sqrt{y^2+1} = e^x - y$$

$$y \pm \sqrt{y^2+1} = e^x$$

To find the inverse function, swap x and y: $x \pm \sqrt{x^2+1} = e^y$

$$\ln\left(x \pm \sqrt{x^2+1}\right) = y$$

$$\text{arsinh}\, x = \ln\left(x \pm \sqrt{x^2+1}\right)$$

Identify, and explain, the error in Matthew's proof.

[2 marks]

Matthew used the "$\pm$" sign instead of just "$+$"

$y^2+1 > y^2 \Rightarrow \sqrt{y^2+1} > y \Rightarrow 0 > y - \sqrt{y^2+1}$

$y - \sqrt{y^2+1} < 0$ but $e^x > 0$ for all x so

$e^x \neq y - \sqrt{y^2+1}$ and therefore we just get

$e^x = y + \sqrt{y^2+1}$

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6 (b) Solve $\ln\left(x + \sqrt{x^2 + 1}\right) = 3$

[1 mark]

$\operatorname{arcsinh} x = 3$

$\Rightarrow x = \sinh 3$

$\Rightarrow x = \frac{1}{2}(e^3 - e^{-3})$

7 Find two invariant points under the transformation given by $\begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}$

[2 marks]

$\begin{pmatrix} 2 & 3 \\ 1 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} \Rightarrow \begin{pmatrix} 2x + 3y \\ x + 4y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$

From this, we obtain two equations: $2x+3y=x$ and $x+4y=y$

$\Rightarrow x = -3y$

Let $y=0$, $x=0$

Let $y=1$, $x=-3$

Two invariant points are $(0,0)$ and $(-3,1)$

Turn over ▶

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8 $2 - 3\text{i}$ is one root of the equation

$$z^3 + mz + 52 = 0$$

where m is real.

8 (a) Find the other roots.

[3 marks]

As the equation has real coefficients, there is another root that is the complex conjugate of $2-3i$ which is $2+3i$

$\sum \alpha = -\frac{b}{a}$

$\Rightarrow \alpha + (2+3i) + (2-3i) = 0$

$\Rightarrow \alpha + 4 = 0 \Rightarrow \alpha = -4$

Do not write outside the box

8 (b) Determine the value of m.

[2 marks]

$z^3 + mz + 52 = 0$

Substituting in $z = -4$

$(-4)^3 + m(-4) + 52 = 0$

$-64 - 4m + 52 = 0$

$-16 - m + 13 = 0$

$m = -3$

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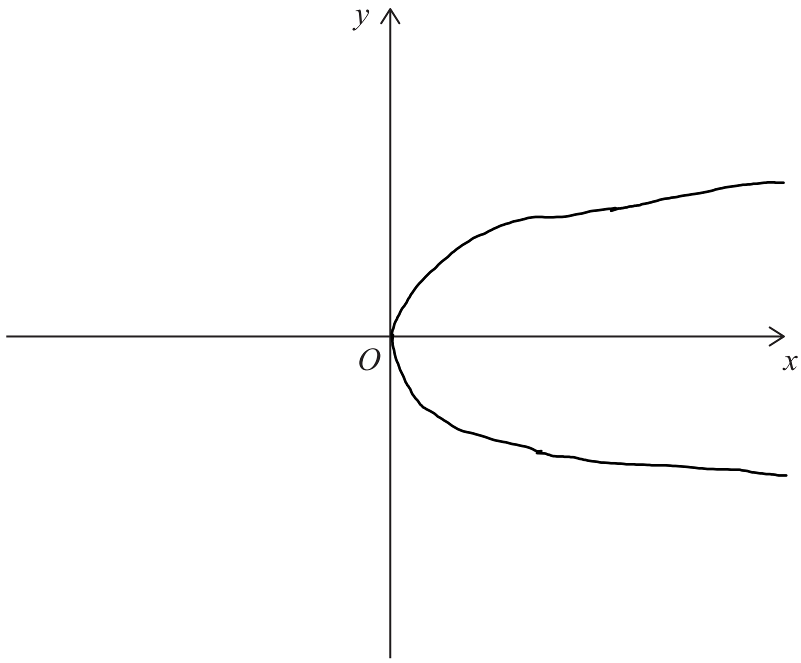
9 (a) Sketch the graph of $y^2 = 4x$

[1 mark]



9 (b) Ben is using a 3D printer to make a plastic bowl which holds exactly 1000 cm^3 of water.

Ben models the bowl as a region which is rotated through 2π radians about the x-axis.

He uses the finite region enclosed by the lines $x = d$ and $y = 0$ and the curve with equation $y^2 = 4x$ for $y \geq 0$

9 (b) (i) Find the depth of the bowl to the nearest millimetre.

[4 marks]

$V = \int_a^b \pi y^2 \, dx$

$\Rightarrow 1000 = \pi \int_0^d 4x \, dx$

$\Rightarrow 250 = \pi \left[\frac{1}{2}x^2\right]_0^d$

$\Rightarrow 250 = \frac{\pi}{2} d^2$

$\Rightarrow \frac{500}{\pi} = d^2 \Rightarrow d = \sqrt{\frac{500}{\pi}}$

depth = 12.6 cm = 126 mm

9 (b) (ii) What assumption has Ben made about the bowl?

[1 mark]

That the thickness of the plastic is negligible

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10 (a) Prove by induction that, for all integers $n \geq 1$,

$$\sum_{r=1}^{n} r^3 = \frac{1}{4}n^2(n+1)^2$$

[4 marks]

Claim: $\sum_{r=1}^{n} r^3 = \frac{1}{4}n^2(n+1)^2 \quad (*)$

Let $n=1$

$\sum_{r=1}^{1} r^3 = 1$ and $\frac{1}{4}(1)^2(1+1)^2 = 1$

Therefore, $(*)$ is true for $n=1$

Assume that for $n=k$, $(*)$ is true such that

$$\sum_{r=1}^{k} r^3 = \frac{1}{4}k^2(k+1)^2$$

Let $n=k+1$

$$\sum_{r=1}^{k+1} r^3 = \sum_{r=1}^{k} r^3 + (k+1)^3$$

$$= \frac{1}{4}k^2(k+1)^2 + (k+1)^3$$

$$= \frac{1}{4}(k+1)^2\left[k^2 + 4(k+1)\right]$$

$$= \frac{1}{4}(k+1)^2\left[k^2+4k+4\right]$$

$$= \frac{1}{4}(k+1)^2(k+2)^2$$

$$= \frac{1}{4}(k+1)^2((k+1)+1)^2$$

So for $n=k+1$, $(*)$ is true

Therefore as $(*)$ is true for $n=1$ and for $n=k+1$ when assumed to be true for $n=k$, by mathematical induction it is true for all integers $n \geq 1$

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10 (b) **Hence** show that

$$\sum_{r=1}^{2n} r(r-1)(r+1) = n(n+1)(2n-1)(2n+1)$$

[4 marks]

$\sum_{r=1}^{2n} r(r-1)(r+1) = \sum_{r=1}^{2n} (r^3 - r)$

$= \sum_{r=1}^{2n} r^3 - \sum_{r=1}^{2n} r$

$= \frac{1}{4}(2n)^2(2n+1)^2 - \frac{1}{2}(2n)(2n+1)$

$= \frac{1}{4}(4n^2)(2n+1)^2 - n(2n+1)$

$= n(2n+1)\left[n(2n+1) - 1\right]$

$= n(2n+1)(2n^2+n-1)$

$= n(2n+1)(2n^2+2n-n-1)$

$= n(2n+1)(2n(n+1)-(n+1))$

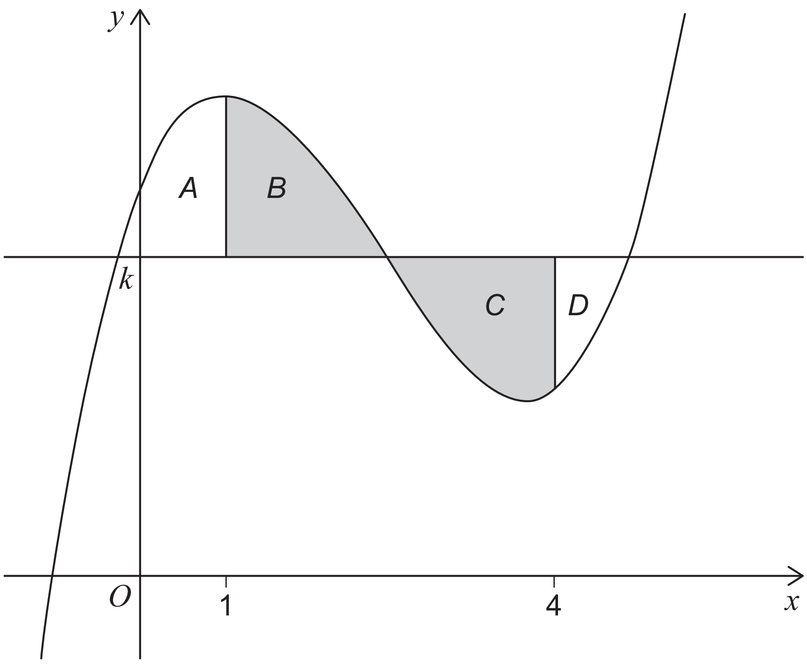
$= n(2n+1)(2n-1)(n+1)$

Do not write outside the box

11 Four finite regions A, B, C and D are enclosed by the curve with equation

$$y = x^3 - 7x^2 + 11x + 6$$

and the lines $y = k$, $x = 1$ and $x = 4$, as shown in the diagram below.



The areas of B and C are equal.

Find the value of k.

[3 marks]

As the areas C and B are equal, the mean value of the function from $x=1$ to $x=4$ is k

$\Rightarrow k = \frac{1}{4-1}\int_1^4 x^3 - 7x^2 + 11x + 6 \, dx$

$= \frac{1}{3}\left[\frac{1}{4}x^4 - \frac{7}{3}x^3 + \frac{11}{2}x^2 + 6x\right]_1^4$

$= \frac{1}{3}\left[\left(\frac{1}{4}(4)^4 - \frac{7}{3}(4)^3 + \frac{11}{2}(4)^2 + 6(4)\right) - \left(\frac{1}{4}(1)^4 - \frac{7}{3}(1)^3 + \frac{11}{2}(1)^2 + 6(1)\right)\right]$

$= \frac{1}{3}\left(\frac{80}{3} - \frac{113}{12}\right)$

$= \frac{23}{4}$

So $k = \frac{23}{4}$

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12 (a) Show that the matrix $\begin{bmatrix} 5-k & 2 \\ k^3+1 & k \end{bmatrix}$ is singular when $k = 1$.

[1 mark]

$\begin{vmatrix} 5-k & 2 \\ k^3+1 & k \end{vmatrix}$ when $k=1$ becomes $\begin{vmatrix} 4 & 2 \\ 2 & 1 \end{vmatrix} = 4(1) - 2(2)$

$= 0$

As the determinant of the matrix is 0, the matrix is singular

12 (b) Find the values of k for which the matrix $\begin{bmatrix} 5-k & 2 \\ k^3+1 & k \end{bmatrix}$ has a negative determinant.

Fully justify your answer.

[5 marks]

$\begin{vmatrix} 5-k & 2 \\ k^3+1 & k \end{vmatrix} = k(5-k) - 2(k^3+1)$

$= 5k - k^2 - 2k^3 - 2 < 0$

$\Rightarrow 2k^3 + k^2 - 5k + 2 > 0$

$2(1)^3 + (1)^2 - 5(1) + 2 = 0$ so $(k-1)$ is a factor of

$2k^3 + k^2 - 5k + 2 = 0$

	$2k^2$	$3k$	-2
k	$2k^3$	$3k^2$	$-2k$
-1	$-2k^2$	$-3k$	2

$\Rightarrow 2k^3 + k^2 - 5k + 2$

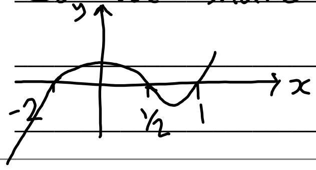
$= (k-1)(2k^2 + 3k - 2)$

$= (k-1)(2k^2 + 4k - k - 2)$

$= (k-1)(2k(k+2) - (k+2))$

$= (k-1)(2k-1)(k+2)$

So we have $(k-1)(2k-1)(k+2) > 0$



so $-2 < k < \frac{1}{2}$ or $k > 1$

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13 The graph of the rational function $y = \mathrm{f}(x)$ intersects the x-axis exactly once at $(-3,\ 0)$

The graph has exactly two asymptotes, $y = 2$ and $x = -1$

13 (a) Find $\mathrm{f}(x)$

[2 marks]

f(x) must be in the form $f(x) = \frac{2x+a}{x+1}$ because $x=-1$ is not defined and gives an asymptote and as x gets large, $y \to 2$ but never reaches 2

Substituting $(-3, 0)$:

$\Rightarrow 0 = \frac{2(-3)+a}{-3+1}$

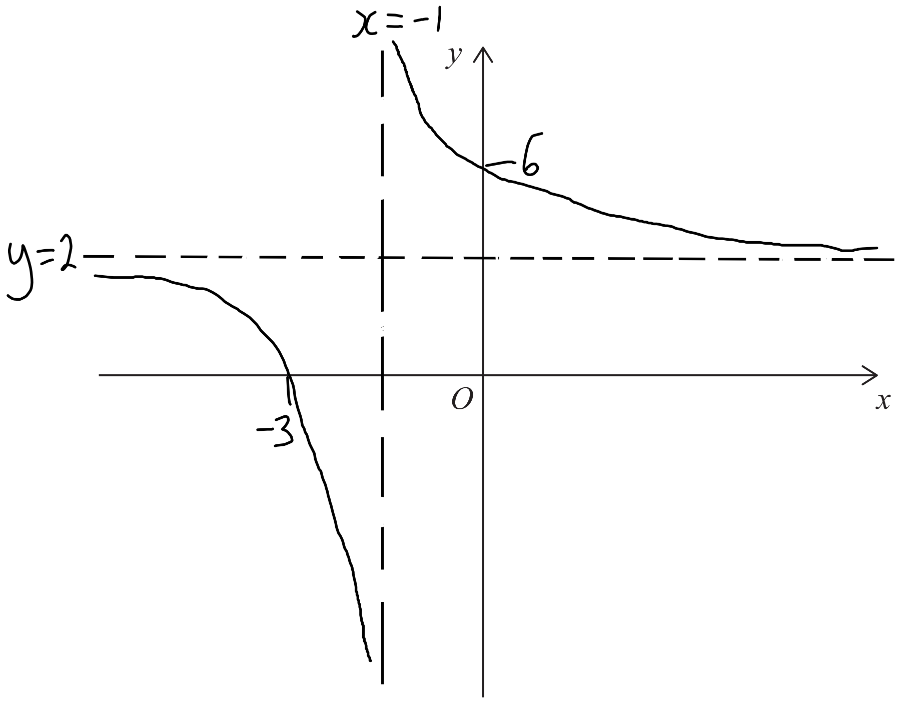
$\Rightarrow 0 = -6+a$

$\Rightarrow a = 6$

$\Rightarrow f(x) = \frac{2x+6}{x+1}$

13 (b) Sketch the graph of the function.

[3 marks]



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13 (c) Find the range of values of x for which $f(x) \le 5$

[4 marks]

$5 = \frac{2x+6}{x+1}$

$5(x+1) = 2x+6$

$5x+5 = 2x+6$

$3x = 1$

$x = \frac{1}{3}$

$\Rightarrow x \geq \frac{1}{3}$

We also get $x < -1$ as the asymptote $y=2$ is the line $y=5$ and this asymptote is at $x=1$

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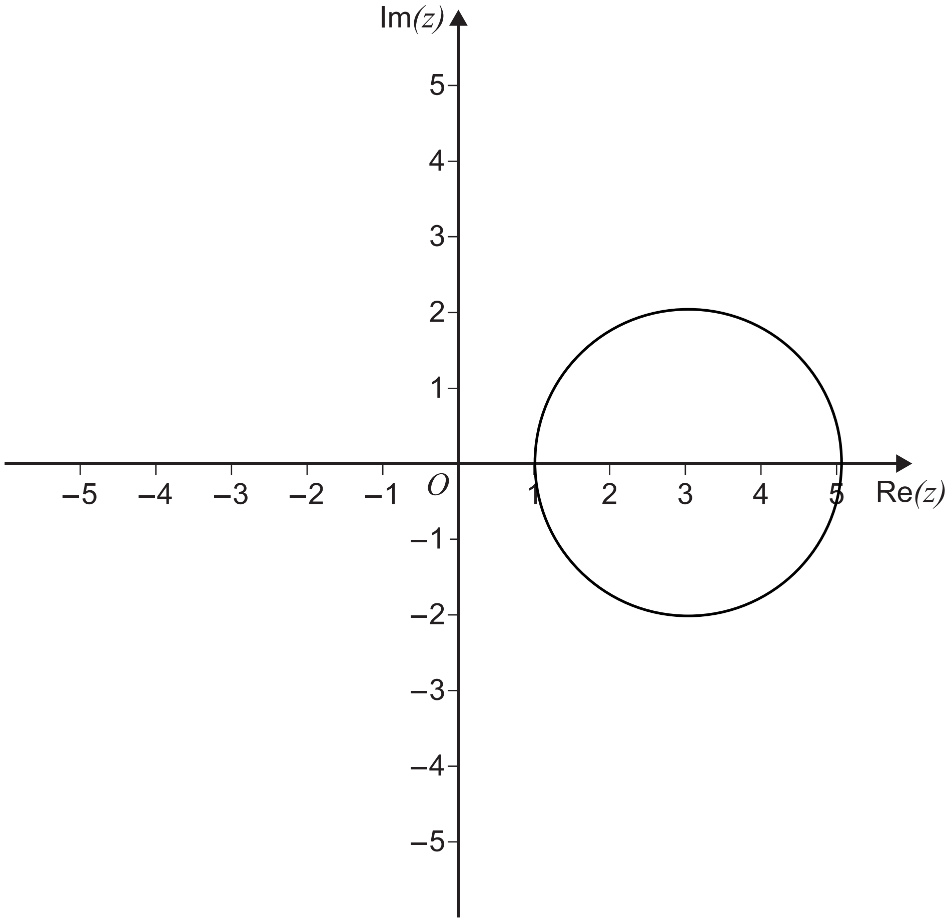
14 (a) Sketch, on the Argand diagram below, the locus of points satisfying the equation

$$|z-3|=2$$

[1 mark]



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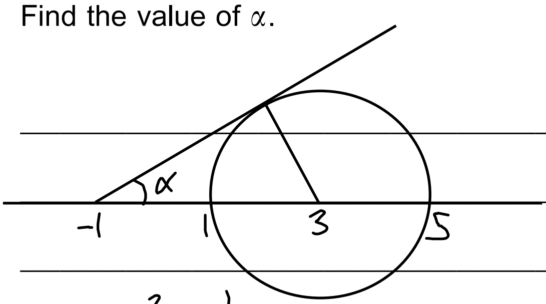
14 (b) There is a unique complex number w that satisfies both

$$|w-3| = 2 \quad \text{and} \quad \arg(w+1) = \alpha$$

where α is a constant such that $0 < \alpha < \pi$

14 (b) (i) Find the value of α.

[2 marks]



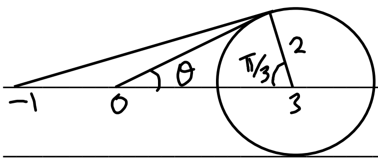
$\sin\alpha = \frac{2}{4} = \frac{1}{2}$

$\alpha = \frac{\pi}{6}$

14 (b) (ii) Express w in the form $r(\cos\theta + \mathrm{i}\sin\theta)$.

Give each of r and θ to two significant figures.

[4 marks]



$r^2 = 2^2 + 3^2 - 2(2)(3)\cos\frac{\pi}{3}$

$\Rightarrow r = \sqrt{7}$

Using the sine rule, we get: $\frac{\sin\theta}{2} = \frac{\sin\frac{\pi}{3}}{\sqrt{7}}$

$\Rightarrow \theta = \arcsin\left(\frac{2}{\sqrt{7}}\sin\frac{\pi}{3}\right)$

$= 0.71$

$w = r(\cos\theta + i\sin\theta)$

$= \sqrt{7}(\cos 0.71 + i\sin 0.71)$

Turn over ▶

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15 (a) Show that

$$\frac{1}{r+2}-\frac{1}{r+3}=\frac{1}{(r+2)(r+3)}$$

[1 mark]

$$\frac{1}{r+2}-\frac{1}{r+3}=\frac{r+3-(r+2)}{(r+2)(r+3)}=\frac{r+3-r-2}{(r+2)(r+3)}=\frac{1}{(r+2)(r+3)}$$

15 (b) Use the method of differences to show that

$$\sum_{r=1}^{n}\frac{1}{(r+2)(r+3)}=\frac{n}{3(n+3)}$$

[3 marks]

$$\sum_{r=1}^{n}\frac{1}{(r+2)(r+3)}=\sum_{r=1}^{n}\frac{1}{r+2}-\frac{1}{r+3}$$

r	$\frac{1}{r+2}$	$-\frac{1}{r+3}$
1	$\frac{1}{3}$	$-\cancel{\frac{1}{4}}$
2	$\cancel{\frac{1}{4}}$	$-\cancel{\frac{1}{5}}$
⋮	⋮	⋮
$n-1$	$\cancel{\frac{1}{n+1}}$	$-\cancel{\frac{1}{n+2}}$
n	$\cancel{\frac{1}{n+2}}$	$-\frac{1}{n+3}$

So $\displaystyle\sum_{r=1}^{n}\frac{1}{(r+2)(r+3)}=\sum_{r=1}^{n}\frac{1}{r+2}-\frac{1}{r+3}=\frac{1}{3}-\frac{1}{n+3}=\frac{n+3-3}{3(n+3)}$

$$=\frac{n}{3(n+3)}$$

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16 Two matrices **A** and **B** satisfy the equation

$$\mathbf{AB} = \boldsymbol{I} + 2\mathbf{A}$$

where $\boldsymbol{I}$ is the identity matrix and $\mathbf{B} = \begin{bmatrix} 3 & -2 \\ -4 & 8 \end{bmatrix}$

Find **A**.

[3 marks]

$AB = I + 2A$

$\Rightarrow AB - 2A = I$

$\Rightarrow A(B - 2I) = I$

$\Rightarrow A = I(B - 2I)^{-1}$

$A = \begin{pmatrix} 1 & -2 \\ -4 & 6 \end{pmatrix}^{-1}$

$= -\frac{1}{2}\begin{pmatrix} 6 & 2 \\ 4 & 1 \end{pmatrix}$

$A = \begin{pmatrix} -3 & -1 \\ -2 & -\frac{1}{2} \end{pmatrix}$

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17 Find the exact solution to the equation

$$\sinh\theta(\sinh\theta + \cosh\theta) = 1$$

[4 marks]

$\sinh\theta(\sinh\theta + \cosh\theta) = 1$

$\sinh\theta(\frac{1}{2}(e^{\theta} - e^{-\theta}) +$

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18 α, β and γ are the real roots of the cubic equation

$$x^3 + mx^2 + nx + 2 = 0$$

By considering $(\alpha - \beta)^2 + (\gamma - \alpha)^2 + (\beta - \gamma)^2$, prove that

$$m^2 \geq 3n$$

[4 marks]

$(\alpha-\beta)^2 + (\gamma-\alpha)^2 + (\beta-\gamma)^2$

$= \alpha^2 - 2\alpha\beta + \beta^2 + \gamma^2 - 2\alpha\gamma + \alpha^2 + \beta^2 - 2\beta\gamma + \gamma^2$

$= 2(\alpha^2+\beta^2+\gamma^2) - 2(\alpha\beta + \alpha\gamma + \beta\gamma)$

$= 2\sum\alpha^2 + 2\sum\alpha\beta$

$= 2\left((\sum\alpha)^2 - 2\sum\alpha\beta\right) - 2\sum\alpha\beta$

$= 2(\sum\alpha)^2 - 6\sum\alpha\beta$

$= 2(-m)^2 - 6(n)$

$= 2m^2 - 6n$

As α, β and γ are real then each of $(\alpha-\beta)^2$ $(\gamma-\alpha)^2$ and $(\beta-\gamma)^2$ are non-negative so their sum is greater than or equal to 0

Therefore, $2m^2 - 6n \geqslant 0$

$m^2 - 3n \geqslant 0$

$m^2 \geqslant 3n$

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19 A theme park has two zip wires.

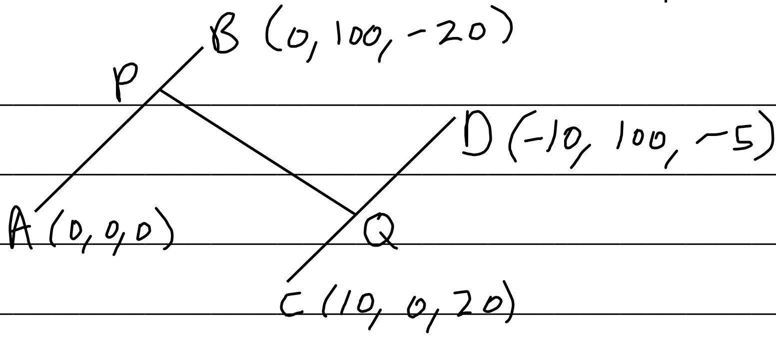
Sarah models the two zip wires as straight lines using coordinates in metres.

The ends of one wire are located at (0, 0, 0) and (0, 100, −20)

The ends of the other wire are located at (10, 0, 20) and (−10, 100, −5)

19 (a) Use Sarah's model to find the shortest distance between the zip wires.

[7 marks]



$AB: \underline{r}_1 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ 100 \\ -20 \end{pmatrix}$

$CD: \underline{r}_2 = \begin{pmatrix} 10 \\ 0 \\ 20 \end{pmatrix} + \mu \begin{pmatrix} 20 \\ -100 \\ 25 \end{pmatrix}$

$\underline{r}_2 - r_1 = \begin{pmatrix} 10 + 20\mu \\ -100\mu - 100\lambda \\ 20 + 25\mu + 20\lambda \end{pmatrix}$ (Represents PQ)

The direction vectors of $\underline{r}_1$ and $\underline{r}_2$ are perpendicular to PQ

$\begin{pmatrix} 10 + 20\mu \\ -100\mu - 100\lambda \\ 20 + 25\mu + 20\lambda \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 100 \\ -20 \end{pmatrix} = 0$ and $\begin{pmatrix} 10 + 20\mu \\ -100\lambda - 100\mu \\ 20 + 25\mu + 20\lambda \end{pmatrix} \cdot \begin{pmatrix} 20 \\ -100 \\ 25 \end{pmatrix} = 0$

$\Rightarrow -10000\mu - 10000\lambda - 400 - 500\mu - 400\lambda = 0$ and

$200 + 400\mu + 10000\mu + 10000\lambda + 500 + 625\mu + 500\lambda = 0$

From the calculator, $\lambda = \frac{2}{3}$ and $\mu = -\frac{44}{63}$

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$|\underline{r}_2 - \underline{r}_1| = \sqrt{(10+29\mu)^2 + (-100\mu - 100\lambda)^2 + (20 + 25\mu + 20\lambda)^2}$

Substituting in our values for μ and λ gives $|PQ| = 16.7$m

19 (b) State one way in which Sarah's model could be refined.

[1 mark]

Take the thickness of the wires into consideration

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